Subject: Calculus
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Volumes Of Solids Of Revolution

Let S be a solid that lies between *x=a* and *x=b.* If the cross-sectional area of S in the plane P_x, through *x* and perpendicular to the *x-axis,* is *A(x)*, where *A* is continuous function, then volume of *S* is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(xi\, *)\Delta x = \int_a^b A(x)dx$$

Finding Volume Using Circular Disks:

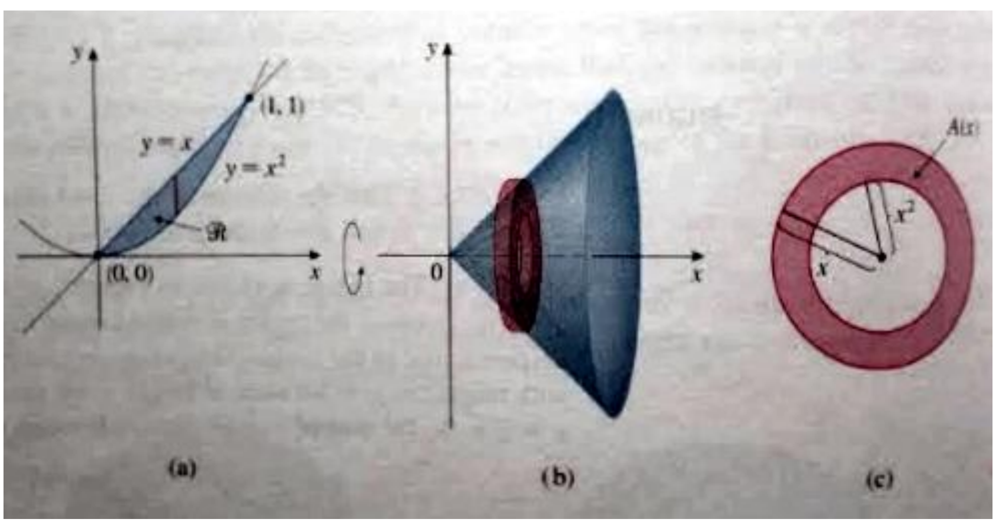
In this method, we evaluate the volume as an integration of multiple disks.

Example1: The region R enclosed by curves *y=x* and *y=x²* is rotated about the *x-axis*. Find the volume of the resulting solid.

Solution: These curves intersect at the coordinates (0,0) and (0,1). The resulting figure of rotation is a *washer* or *annular ring* with inner radius x^2 and outer radius *x* (Figures a and b). Therefore we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle.

Step 1: We need to find *A(x)* , which we integrate to find the volume V.
The area of a disk is given by $A = \pi r^2$. For a ring we have our outer and inner radii as *x and* x^2 respectively. Therefore the area of the resulting ring is given by (See figure c):

$$A(x) = \pi(outer\ radius)^2 - \pi(\text{inner radius})^2$$



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$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

Step 2: To find the volume of the figure, we integrate the area *A*(*x*) from 0 to 1.

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^2 - x^4)dx$$

$$= \pi[\frac{x^3}{3} - \frac{x^5}{5}]_0^1 = \frac{2\pi}{15}$$

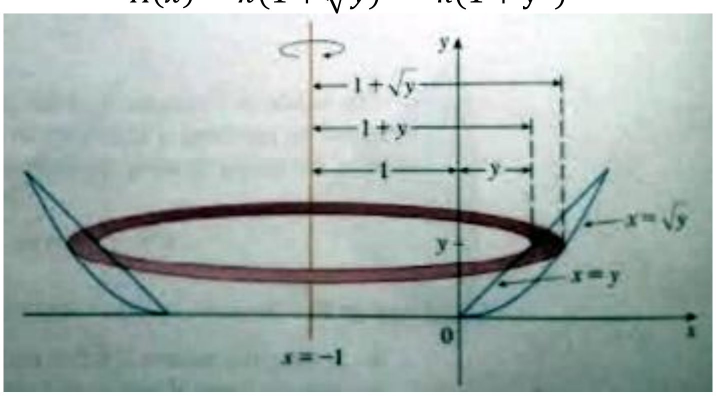
Therefore the volume of the resulting figure is $\frac{2\pi}{15}$ cubic units.

Example 2: Find the volume of the solid obtained by rotating the curves from the previous example about the line *x = -1*.

Solution: The horizontal cross section of the resulting solid of revolution is given in Figure (a).
Step 1: The figure is a washer with inner radius *1+y* and outer radius $1 + \sqrt{y}$ so the cross sectional area is

$$A(y) = \pi(outer\ radius)^2 - \pi(\text{inner radius})^2$$

$$A(x) = \pi(1 + \sqrt{y})^2 - \pi(1 + y^2)^2$$



Step 2: The volume is given by:

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$$V = \int_0^1 A(y)dy = [\pi\left(1+\sqrt{y}\right)^2 - \pi(1+y^2)^2]dy$$

$$= \pi[\frac{4y^{\frac{3}{2}}}{3} - \frac{y^2}{2} - \frac{y^3}{3}]_0^1 = \frac{\pi}{2}$$

Therefore the volume of the resulting figure is $\frac{\pi}{2}$ cubic units.

Finding Volume Using Cylindrical Shells:

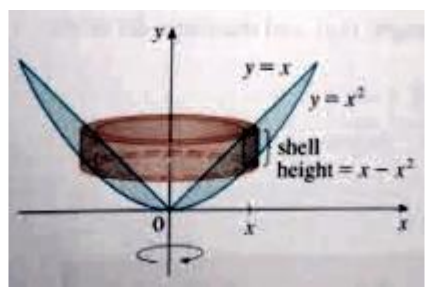
This method is similar to the disks but instead we evaluate the volume as an integration of multiple cylinders instead of disks. Therefore the collective volume is given by:

$\int_a^b 2\pi x * f(x)dx$ where $2\pi x$ is the circumference, f(x) is the height and dx is the thickness.

Example 3: Find the volume of the solid obtained by rotating about the y-axis the region between y=x and y=x^2.

Solution:

Step 1: The region described is a typical shell as shown in the figure. The shell has radius x, circumference $2\pi x$, and height x-x^2.



Step 2:

The volume is given by

$$V = \int_0^1 2\pi x(x - x^2)dx$$

$$= 2\pi[\frac{x^3}{3} - \frac{x^4}{4}]_0^1 = \frac{\pi}{6}$$

Therefore the volume of the resulting figure is $\frac{\pi}{6}$ cubic units.

Reference: *Stewart James, Calculus Early Transcendentals, 8th Ed.*