

Optimization Problems

Overview: The methods used for finding extreme values for functions have practical applications in many areas of life. For example, a traveler wanting to minimize transportation time. In solving such practical problems, one has to convert the word problem into a mathematical optimization problem and set up a function to be maximized or minimized.

Steps in solving optimization problems:

- 1. Understand the problem:** Read the problem carefully to find out what the problem is asking. Then, underline the important pieces of information in the problem.
- 2. Draw a diagram (optional):** It is always helpful to sum up the entire problem in a simple diagram so to prevent reading the problem repeatedly.
- 3. Introduce notion and express it in terms of variables:** Assign a symbol to the quantity that has to be maximized or minimized. (Let us call it Z for now). Assign variable names to the unknowns and express Z as a function of those variables.
- 4. Try to express notion in terms of one variable:** If Z has been expressed as a function of more than one variable (step 4), use the information in the problem to eliminate all but one of the variables and use that to express Z .
- 5. Differentiate function and equate it to zero to obtain critical points:** Differentiate Z with respect to the variable you choose and equate it to zero to obtain the values of that variable (critical points).
- 6. Test critical points for max/min using the second derivative:** To test whether the critical points are a max or min (concave down or up respectively), we take the second derivative of Z and plug in the critical points obtained in step 6 to see whether we get a positive value (minimum) or a negative value (maximum.)
- 7. Use the required critical point to find the optimal answer:** Once we know what critical points we are using, we plug that in Z to obtain the answer to our problem.

Now let us use these steps on a few examples!



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Problem 1: The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

SOLUTION:

STEP 1: We have,

- a) Sum of two positive numbers is 16 (**info**)
- b) **Smallest** possible value for sum of their squares (**objective**)

STEP 2 will be ignored since diagram is not possible for this problem.

STEP 3: Let us assume the numbers are x and y and the sum of their squares is S .

We are **given:** $x + y = 16$ $x > 0$ and $y > 0$ (since they are **positive**)

We need to **minimize:** $S = x^2 + y^2$

STEP 4: Since S is expressed in terms of both x and y , we need to try to **express S with only one variable**. From the information given, we have $x + y = 16$, or $y = 16 - x$. ----- (1)

Putting this value of y in S we get,

$S(x) = x^2 + (16-x)^2$. When we simplify this, $S(x) = x^2 + x^2 - 24x + 256$. Combining like terms we get, $S(x) = 2x^2 - 32x + 256$ which we need to **minimize**.

STEP 5: Calculating the first derivative of $S(x)$ we get,

$$S'(x) = 4x - 32$$

To find the critical points, we need to equate $S'(x)$ to zero.

$$\Rightarrow 4x - 32 = 0 \text{ or } 4x = 32 \text{ or } x = 8. \text{ ----- (2)}$$

STEP 6: To test whether this value gives us a minimum, we take out the second derivative:

$S''(x) = 4$, which is **positive**. Thus, we know that we have **minimized S** .

STEP 7: But, we still haven't completed the problem. We still need to find the other number, i.e. y .

From (1), we have $y = 16 - x$.

After putting in the $x = 8$ we found from (2) and solving for y , we get $y = 8$.

Therefore, the two positive numbers, which add up to 16 and, with the smallest possible value for the sum of their squares, are 8 and 8.



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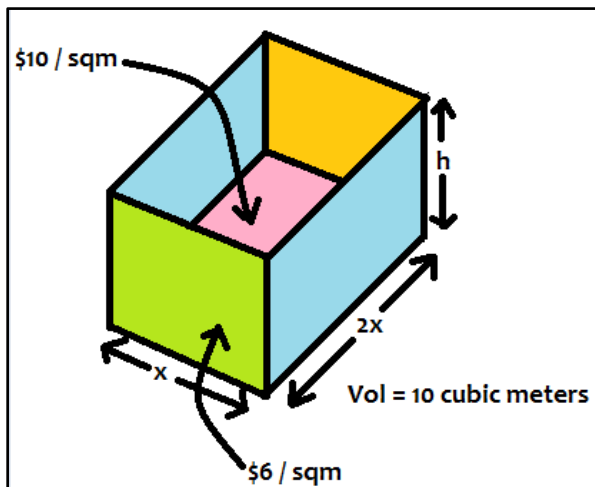
Problem 2: A rectangular storage container with an open top needs to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs $\$10$ per m^2 . Material for the sides costs $\$6$ per m^2 . Find the cost of the material for the cheapest container.

SOLUTION:

STEP 1: We have,

- Rectangular box with open top (**info**)
- Volume of the box is 10 m^3 (**info**)
- Length of the base is twice its width (**info**)
- Material for the base costs $\$10 / \text{m}^2$ and material for the sides costs $\$6 / \text{m}^2$ (**info**)
- The cost of the material for the **cheapest** container (**objective**)

STEP 2:



STEP 3: Let us assume the height of the box is h and the width of the box is x . Then from the information given, we obtain the length of the box as $2x$. Let the cost of the box be C .

Cost = $\$10(\text{area of base}) + \$6(\text{area of 2 long sides}) + \$6(\text{area of 2 short sides})$

which gives,

$$\begin{aligned} \text{Cost} &= (10 * x * 2x) + (2 * 6 * x * h) + (2 * 6 * 2x * h) \\ &= 20x^2 + 12xh + 24xh \\ &= 20x^2 + 36xh \end{aligned}$$

STEP 4: Since C is expressed in terms of both x and h , we need to try to **express S with only one variable**. From the information given, we have the volume of the box = 10 m^3 , which means:

$$x \cdot 2x \cdot h = 10 \text{ or } h = \frac{10}{2x^2}$$

Putting this value of h in C we get,

$C(x) = 20x^2 + \frac{36 \cdot x \cdot 10}{2x^2}$. When we simplify this, $C(x) = 20x^2 + 180x^{-1}$, which we need to **minimize**.



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STEP 5: Calculating the first derivative of $C(x)$ we get,

$$C'(x) = 40x = 180x^{-2}$$

To find the critical points, we need to equate $S'(x)$ to zero.

$$\Rightarrow 40x - 180x^{-2} = 0$$

$$\Rightarrow 40x = \frac{180}{x^2}$$

$$\Rightarrow 40x^3 = 180$$

$$\Rightarrow x^3 = \frac{180}{40} \text{ or } x^3 = \frac{9}{2}$$

This will give us $x = \left(\frac{9}{2}\right)^{1/3}$. Solving this using a calculator we get $x \approx 1.65$

STEP 6: To test whether this value gives us a minimum, we take out the second derivative:

$C''(x) = 40 + \frac{360}{x^3}$. If we put $x = 1.65$ in this, we get $C''(x) = 40 + \frac{360}{1.65^3}$. Solving this using a calculator, we get $C''(x) = 120.14$, which is **positive**. Thus, we know that we have **minimized C**.

STEP 7: But, we still haven't completed the problem. We still need to find the cost of the material for the cheapest container.

After putting in $x = 1.65$ in our original cost function we get: $C(1.65) = 20(1.65)^2 + 180(1.65)^{-1}$

Using a calculator, we obtain the cost as **C \approx \$163.54**

Therefore, the cost of the material for the cheapest container given the dimensions is \$163.54

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References: The following works were referred to during the creation of this handout: [Khan's Academy's "Optimization: Sum of Squares"](#), [Khan Academy's "Optimization: Cost of Materials"](#), and *Stewart Calculus: Early Transcendentals, 5th Edition*.



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