Subject: Other Math
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Optimization Problems

Overview: The methods used for finding extreme values for functions have practical applications in many areas of life. For example, a traveler wanting to minimize transportation time. In solving such practical problems, one has to convert the word problem into a mathematical optimization problem and set up a function to be maximized or minimized.

Steps in solving optimization problems:

1. **Understand the problem:** Read the problem carefully to find out what the problem is asking. Then, underline the important pieces of information in the problem.
2. **Draw a diagram (optional):** It is always helpful to sum up the entire problem in a simple diagram so to prevent reading the problem repeatedly.
3. **Introduce notion and express it in terms of variables:** Assign a symbol to the quantity that has to be maximized or minimized. (Let us call it Z for now). Assign variable names to the unknowns and express Z as a function of those variables.
4. **Try to express notion in terms of one variable:** If Z has been expressed as a function of more than one variable (step 4), use the information in the problem to eliminate all but one of the variables and use that to express Z.
5. **Differentiate function and equate it to zero to obtain critical points:** Differentiate Z with respect to the variable you choose and equate it to zero to obtain the values of that variable (critical points).
6. **Test critical points for max/min using the second derivative:** To test whether the critical points are a max or min (concave down or up respectively), we take the second derivative of Z and plug in the critical points obtained in step 6 to see whether we get a positive value (minimum) or a negative value (maximum.)
7. **Use the required critical point to find the optimal answer:** Once we know what critical points we are using, we plug that in Z to obtain the answer to our problem.

Now let us use these steps on a few examples!

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Problem 1: ***The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?***

SOLUTION:

STEP 1: We have,
a) Sum of two positive numbers is 16 **(info)**
b) **Smallest** possible value for sum of their squares **(objective)**

STEP 2 will be ignored since diagram is not possible for this problem.

STEP 3: Let us assume the numbers are **x** and **y** and the sum of their squares is **S**.

We are **given:** $x + y = 16$ $x > 0$ and $y > 0$ (since they are **positive**)
We need to **minimize:** $S = x^2 + y^2$

STEP 4: Since S is expressed in terms of both x and y, we need to try to **express S with only one variable**. From the information given, we have $x + y = 16$, or $y = 16 - x$. ----------- (1)
Putting this value of **y** in **S** we get,
$S(x) = x^2 + (16-x)^2$. When we simplify this, $S(x) = x^2 + x^2 - 24x + 256$. Combining like terms we get,
$S(x) = 2x^2 - 32x + 256$ which we need to **minimize**.

STEP 5: Calculating the first derivative of S(x) we get,
$S'(x) = 4x - 32$
To find the critical points, we need to equate S'(x) to zero.
=> $4x - 32 = 0$ or $4x = 32$ or $x = 8$. ----------- (2)

STEP 6: To test whether this value gives us a minimum, we take out the second derivative:
$S''(x) = 4$, which is **positive.** Thus, we know that we have **minimized S**.

STEP 7: But, we still haven't completed the problem. We still need to find the other number, i.e. **y**.
From (1), we have $y = 16 - x$.

After putting in the $x = 8$ we found from (2) and solving for y,
we get $y = 8$.

Therefore, the two positive numbers, which add up to 16 and, with the smallest possible value for the sum of their squares, are 8 and 8.

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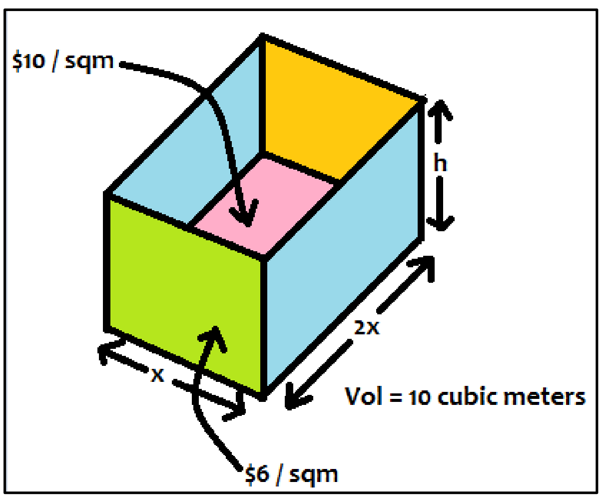
Problem 2: ***A rectangular storage container with an open top needs to have a volume of 10 m³. The length of its base is twice the width. Material for the base costs $10 per m². Material for the sides costs $6 per m². Find the cost of the material for the cheapest container.***

SOLUTION:

STEP 1: We have,
a) Rectangular box with open top **(info)**
b) Volume of the box is 10 m^3 **(info)**
c) Length of the base is twice its width **(info)**
d) Material for the base costs $10 / m^2 and material for the sides costs $6 / m^2 **(info)**
e) The cost of the material for the **cheapest** container **(objective)**

STEP 2:



STEP 3: Let us assume the height of the box is **h** and the width of the box is **x.** Then from the information given, we obtain the length of the box as **2x.** Let the cost of the box be **C.**

Cost = $10(area of base) + $6(area of 2 long sides) + $6(area of 2 short sides)
which gives,
Cost= $(10 * x * 2x) + (2 * 6 * x * h) + (2 * 6 * 2x * h)$
$= 20x^2 + 12xh + 24xh$
$= 20x^2 + 36xh$

STEP 4: Since C is expressed in terms of both x and h, we need to try to **express S with only one variable.** From the information given, we have the volume of the box = 10m^3, which means:
x.2x.h= 10 or $h = \frac{10}{2x^2}$.
Putting this value of **h** in **C** we get,
C(x) = $20x^2 + \frac{36.x.10}{2x^2}$.When we simplify this, **C(x) = $20x^2 + 180x^{-1}$** , which we need to **minimize.**

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STEP 5: Calculating the first derivative of C(x) we get,
$C'(x) = 40x = 180x^{-2}$
To find the critical points, we need to equate S'(x) to zero.
=> $40x - 180x^{-2} = 0$
=> $40x = \frac{180}{x^2}$
=> $40x^3 = 180$
=> $x^3 = \frac{180}{40}$ or $x^3 = \frac{9}{2}$
This will give us $x = \left(\frac{9}{2}\right)^{1/3}$. Solving this using a calculator we get **x ≈ 1.65**

STEP 6: To test whether this value gives us a minimum, we take out the second derivative:
$C''(x) = 40 + \frac{360}{x^3}$. If we put x = 1.65 in this, we get $C''(x) = 40 + \frac{360}{1.65^3}$. Solving this using a calculator, we get C''(x) = 120.14, which is **positive.** Thus, we know that we have **minimized C**.

STEP 7: But, we still haven't completed the problem. We still need to find the cost of the material for the cheapest container.

After putting in x = 1.65 in our original cost function we get: $C(1.65) = 20(1.65)^2 + 180(1.65)^{-1}$
Using a calculator, we obtain the cost as **C ≈ $163.54**

Therefore, the cost of the material for the cheapest container given the dimensions is $163.54

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References: The following works were referred to during the creation of this handout: Khan's Academy's "Optimization: Sum of Squares", Khan Academy's "Optimization: Cost of Materials", and *Stewart Calculus: Early Transcendentals, 5th Edition.*