Integration By Parts

Integration by parts is a technique for performing indefinite integration $\int u\,dv$ or definite integration $\int_a^b u\,dv$ by expanding the differential of a product of functions $d(uv)$ and expressing the original integral in terms of a known integral $\int v\,du$. A single integration by parts starts with

$$d(uv) = u\,dv + v\,du, \tag{1}$$

and integrates both sides,

$$\int d(uv) = uv = \int u\,dv + \int v\,du. \tag{2}$$

Rearranging gives

$$\int u\,dv = uv - \int v\,du. \tag{3}$$

For example, consider the integral $\int x\cos x\,dx$ and let

$$u = x \quad dv = \cos x\,dx \tag{4}$$
$$du = dx \quad v = \sin x, \tag{5}$$

so integration by parts gives

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx \tag{6}$$
$$= x\sin x + \cos x + C, \tag{7}$$

where C is a constant of integration.
The procedure does not always succeed, since some choices of u may lead to more complicated integrals than the original. For example, consider again the integral $\int x\cos x\,dx$ and let

$$u = \cos x \quad dv = x\,dx$$
$$du = -\sin x\,dx \quad v = \tfrac{1}{2}x^2, \tag{8}$$

Integration By Parts

giving

$$\int x\cos x\,dx = \tfrac{1}{2}x^2\cos x - \tfrac{1}{2}\int x^2(-\sin x)\,dx \tag{9}$$

$$= \tfrac{1}{2}x^2\cos x + \tfrac{1}{2}\int x^2\sin x\,dx, \tag{10}$$

which is more difficult than the original (Apostol 1967, pp. 218-219).
Integration by parts may also fail because it leads back to the original integral. For example, consider $\int x^{-1}\,dx$ and let

$$\begin{aligned} u &= x \quad dv = x^{-2}\,dx \\ du &= dx \quad v = -x^{-1}, \end{aligned} \tag{11}$$

then

$$\int x^{-1}\,dx = -1 - \int(-x^{-1})\,dx + C = \int x^{-1}\,dx + C - 1, \tag{12}$$

which is same integral as the original (Apostol 1967, p. 219).

The analogous procedure works for definite integration by parts, so

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du, \tag{13}$$

where $[f]_a^b = f(b) - f(a)$.
Problems-

PROBLEM 1 : Integrate $\int xe^x\,dx$

PROBLEM 2 : Integrate $\int x\sin x\,dx$

PROBLEM 3 : Integrate $\int x\ln x\,dx$

PROBLEM 4 : Integrate $\int x\cos 3x\,dx$

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References: The following works were referred to during the creation of this handout: D.A. Kouba's *Integration by Parts.*