Chain Rule

Background

The chain rule allows us to take the derivative of a composite function, meaning \( f(g(x)) \). For example, if \( y = \sin(\ln(x)) \) the outer function \( f(x) \) is \( \sin(x) \) and the inner function \( g(x) \) is \( \ln(x) \). Identifying the inner and outer functions is a crucial step in finding the derivative of a composite function. The following are some more examples:

- \( y = \ln(x^2) \): outer function \( f(x) = \ln(x) \) and inner function \( g(x) = x^2 \)
- \( y = e^{2x} \): outer function \( f(x) = e^x \) and inner function \( g(x) = 2x \)
- \( y = (x^4 - 5x)^3 \): outer function \( f(x) = x^3 \) and inner function \( g(x) = x^4 - 5x \)

You try!

Find the outer and inner functions for the following:

1. \( y = \tan^3(x) \) [Hint: \( \tan^3(x) = (\tan(x))^3 \)]
2. \( y = \log(3x^5 - e^x) \)
3. \( y = 2^{4x^5} \)

Definition

Now that we have an understanding of composite functions, we can move on to the chain rule definition. The chain rule says that for a composite function \( y = f(g(x)) \), the derivative is

\[ y' = f'(g(x)) \cdot g'(x) \]

For example, if \( y = \sin(\ln(x)) \), then \( y' = \cos(\ln(x)) \cdot 1/x = \cos(\ln(x))/x \). Here are some more examples:

- \( y = \ln(x^2), \ y' = (1/x^2) \cdot d/dx(x^2) = (1/x^2) \cdot (2x) = 2x/x^2 = 2/x \)
- \( y = e^{2x}, \ y' = e^{2x} \cdot d/dx(2x) = e^{2x} \cdot 2 = 2e^{2x} \)
- \( y = (x^4 - 5x)^3, \ y' = 3(x^4 - 5x)^2 \cdot d/dx(x^4 - 5x) = 3(x^4 - 5x)^2(4x^3 - 5) \)
Chain Rule

You try!

Find the derivatives of the following:

1. \( y = \tan^3(x) \) [Hint: \( \tan^3(x) = (\tan(x))^3 \)]
2. \( y = \log(3x^5 - e^x) \)
3. \( y = 2^{4x^5} \)

Chain Rule With Complex Composite Functions

The chain rule can also be applied with more complex composite functions. For example, \( y = \ln(\sin(e^{2x})) \) is a composite function comprised of four functions: \( \ln(x) \), \( \sin(x) \), \( e^x \), and \( 2x \). To find the derivative, apply the chain rule starting from the outer most function which in this case is \( \ln(x) \).

\[
y = \ln(\sin(e^{2x}))
\]
\[
y' = \left[1/(\sin(e^{2x}))\right] \cdot \frac{d}{dx}(\sin(e^{2x}))
\]
\[
= \left[1/(\sin(e^{2x}))\right] \cdot \cos(e^{2x}) \cdot \frac{d}{dx}(e^{2x})
\]
\[
= \left[1/(\sin(e^{2x}))\right] \cdot \cos(e^{2x}) \cdot (e^{2x}) \cdot \frac{d}{dx}(2x)
\]
\[
= \left[1/(\sin(e^{2x}))\right] \cdot \cos(e^{2x}) \cdot (e^{2x}) \cdot 2
\]
\[
= \left[2 \cdot e^{2x} \cdot \cos(e^{2x})\right]/\sin(e^{2x})
\]

You try!

Find the derivatives of the following:

1. \( y = \cos^5(4x^5) \) [Hint: \( \cos^5(x) = (\cos(x))^5 \)]
2. \( y = (\log(e^{3x}))^5 \)
3. \( y = \cot(6x^5 - 3x^2 + 5)^4 \)

Reference: Khan Academy