Subject: Pre-Calculus, Calculus 2, and Differential Equations
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Partial Fraction Decomposition

Partial-fraction decomposition is the process of starting with the simplified answer and taking it back apart, of "decomposing" the final expression into its initial polynomial fractions. There are four distinct cases that are explained within this handout.

Before starting, consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express *f* as a sum of simpler fractions **provided** that **the degree of P is less than the degree of Q.**

Case I: The Denominator Q(x) is a Product of Distinct Linear Factors.

Ex. $f(x) = \frac{x^2+2x\text{-}1}{2x^3+3x^2\text{-}2x}$ (notice that the lead power is bigger for the denominator Q than for P)

$Q(x) = 2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$

Since the denominator has three distinct factors, the partial fraction decomposition of the *f*(x) has the form

$$\frac{x^2 + 2x\text{-}1}{x(2x\text{-}1)(x+2)} = \frac{A}{x} + \frac{B}{2x\text{-}1} + \frac{C}{x+2}$$

We then multiply on both sides by the common denominator x(2x-1)(x+2). We then get:

$$x^2 + 2x\text{-}1 = A(2x\text{-}1)(x+2) + B(x)(x+2) + C(x)(2x\text{-}1)$$

Distribute and simplify:

$$x^2 + 2x\text{-}1 = A(2x^2 + 3x\text{-}2) + B(x^2 + 2x) + C(2x^2\text{-}x)$$

$$x^2 + 2x\text{-}1 = 2Ax^2 + 3Ax\text{-}2A + Bx^2 + 2Bx + 2Cx^2\text{-}Cx$$

Group like terms on the left side:

$$x^2 + 2x\text{-}1 = 2Ax^2 + Bx^2 + 2Cx^2 + 3Ax + 2Bx\text{-}Cx\text{-}2A$$

$$x^2 + 2x\text{-}1 = (2A + B + 2C)x^2 + (3A + 2B\text{-}C)x\text{-}2A$$

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The coefficients of the terms on the left should equal those of similar terms on the right.

$2A + B + 2C = 1$

$3A + 2B - C = 2$

$-2A = -1$

We then solve the linear system of equations by any method which should give us:

A = 1/2 B = 1/5 C = -1/10

We can now rewrite the original irrational fraction as

$$\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}$$

Case II: Q(x) is a Product of Linear Factors, Some Repeated Factors

Ex. $f(x) = \frac{4x}{x^3 - x^2 - x + 1}$

We factor the bottom to get:

$$\frac{4x}{(x-1)^2(x+1)}$$

We can now equate this to its partial fraction form.

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

From here, we can follow the first example to solve this example and the results should be:

A=1, B=2, C=-1

We can rewrite the original function like so:

$$\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

Partial Fraction Decomposition

Case III: Q(x) Contains Irreducible Quadratic Factors, None Repeated

Ex. $f(x)=\dfrac{2x^2-x+4}{x^3+4x}$

The function with the factored bottom looks like so:

$$\frac{2x^2-x+4}{x(x^2+4)}$$

We can now equate this to its partial fraction form.

$$\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

From here, we can also follow example 1 to solve the problem and we should get:

A=1, B=1, C=-1

We can rewrite the original function like so:

$$\frac{1}{x}+\frac{x-1}{x^2+4}$$

Case IV: Q(x) Contains a Repeated Irreducible Quadratic Factor

Ex. $f(x)=\dfrac{x^4-x^3+2x^2+x+1}{x^5+2x^3+x}$

The function with the factored bottom looks like:

$$\frac{x^4-x^3+2x^2+x+1}{x(x^2+1)^2}$$

We can now equate this to its partial fraction form.

$$\frac{x^4-x^3+2x^2+x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Partial Fraction Decomposition

From here, we can again follow example 1 to solve the problem and the results should be:

A=1, B=0, C=-1, D=0, E=2

We can rewrite the original function like so:

$$\frac{1}{x} - \frac{1}{(x^2+1)} + \frac{2}{(x^2+1)^2}$$

Reference: The following works were referred to during the creation of this handout: purplemath.com and *Stewart Calculus: Early Transcendentals 7th edition*.