Limits to Infinity

Summary:

- \( \lim_{x \to \infty} \frac{1}{x} = 0 \)
- \( \lim_{x \to \pm\infty} \frac{\text{power of } x}{\text{same power of } x} = \frac{\text{coefficient of largest } x}{\text{coefficient of largest } x} \)
- \( \lim_{x \to \pm\infty} \frac{\text{smaller power of } x}{\text{larger power of } x} = 0 \)
- \( \lim_{x \to \pm\infty} \frac{\text{larger power of } x}{\text{smaller power of } x} = \pm\infty \)

- Check every term for a higher power of \( x \)—most exams have at least one trick question to make sure that you are paying attention!
- Double-check whether your limit is to \( \infty \) or to \( -\infty \).

You will see this everywhere: \( \lim_{x \to \infty} \frac{1}{x} = 0 \).
It is important to remember this. Similarly, \( \frac{8}{x} = 8 \cdot 0 = 0 \) and \( \frac{1}{x} \cdot \frac{1}{x} = 0 \cdot 0 = 0 \).

Common Cases:

- Both top and bottom have the same highest power of \( x \):
  \[ \lim_{x \to \infty} \frac{x^3 + x}{3x^3} = \frac{x^3 + x}{3x^3} = \frac{3^3 + 3}{3x^3} = \frac{3^3}{3} = \frac{x^3}{x^3} = \frac{1}{3} = \lim_{x \to \infty} \frac{1}{x^3} \]

*Be careful! \( x^5 + x^2 + x^4 = x^4 + x^2 + x + 5 \) The highest power of \( x \) is not always the first.

- The top has a higher power:
  \[ \lim_{x \to \infty} \frac{x^4 + 6}{x^3 - 4} = \frac{x^4 + 6}{x^3 - 4} = \frac{x^4 + \frac{6}{x^3}}{x^3 - \frac{4}{x^3}} = \frac{x^4}{x^3} = \frac{x^0 + 0}{1 - \frac{4}{x^3}} \to \lim_{x \to \infty} x = -\infty \]

- The bottom has a higher power of \( x \):
  \[ \lim_{x \to \infty} \frac{x^3 - x + 2}{7x^5 - 3} = \frac{x^3 - x + 2}{7x^5 - 3} = \frac{x^3 - \frac{x}{x^5} + \frac{2}{x^5}}{7x^5 - \frac{3}{x^5}} = \frac{1}{x^2 - \frac{1}{x^7} + \frac{2}{x^5}} \to \lim_{x \to \infty} \frac{1}{7x^2} = 0 \]

(Turn Over)
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- With square roots:

$$\lim_{x \to \infty} \frac{2x^4 - 7}{\sqrt{4x^8 + 7x^5}} \rightarrow \frac{2x^4 - 7}{\sqrt{4x^8 + 7x^5}} \cdot \frac{\frac{1}{x^4}}{\frac{1}{\sqrt{x^8}} = \frac{2x^4 - 7}{\frac{x^4}{\sqrt{x^8} + \frac{7x^5}{x^5}}} = \frac{2 - 0}{\sqrt{4} + 0} = \frac{2}{2} = 1$$