Subject: Statistics
Created by: Flora Li
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Probability

1. Properties of Probability

(1) Terminology

- $\emptyset$ denotes the null or empty set;
- $A \subset B$ means A is a subset of B;
- $A \cup B$ is the union of A and B;
- $A \cap B$ is the intersection of A and B;
- A' is the complement of A (i.e., all elements in S that are not in A).
- Mutually exclusive and Exhaustive events:

$A_1, A_2, \ldots, A_k$ are mutually exclusive events means that $A_i \cap A_j = \emptyset$, $i \neq j$; that is, $A_1, A_2, \ldots, A_k$ are disjoint sets;

$A_1, A_2, \ldots, A_k$ are exhaustive events means that $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

So if $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive events, we know that $A_i \cap A_j = \emptyset$, $i \neq j$, and $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

(2) Probability Laws

Commutative Laws
$A \cup B = B \cup A$
$A \cap B = B \cap A$

Associative Laws
$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Laws
$(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$

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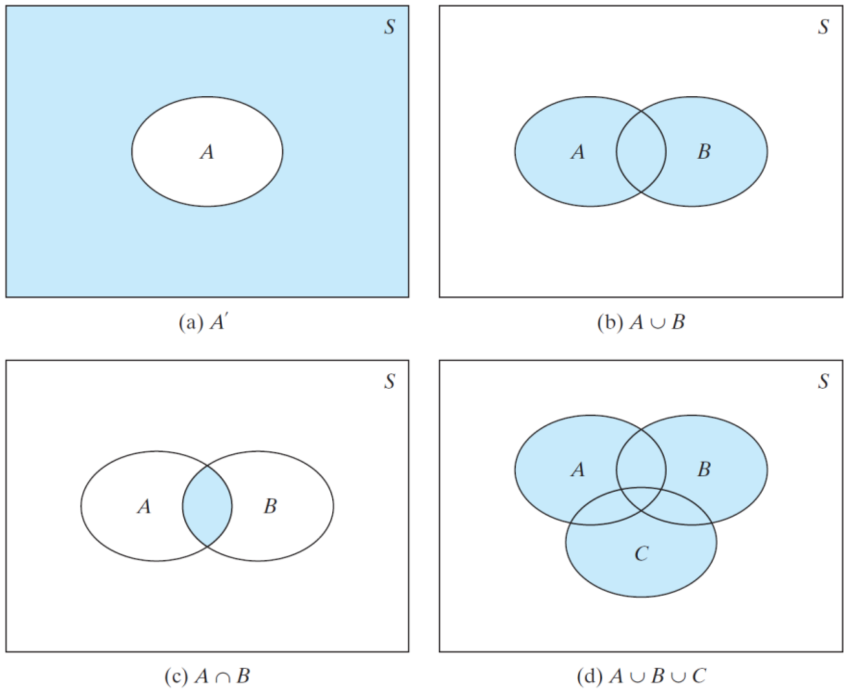
(3) Venn diagrams



(4) Definition

Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number P(A), called the probability of the event A, such that the following properties are satisfied:

a) $P(A) \geq 0$;
b) $P(S) = 1$;
c) if $A_1, A_2, A_3 \dots$ are events and $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup A_3 \cdots = P(A_1) + P(A_2) + P(A_3) + \cdots$$

for an infinite, but countable, number of events.

(5) Theorems

a) For each event A, $P(A) = 1 - P(A')$.
b) $P(\emptyset) = 0$.
c) If events A and B are such that $A \subset B$, then $P(A) \leq P(B)$.

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d) For each event A, P(A) ≤ 1.
e) If A and B are any two events, then P(A ∪ B) = P(A) + P(B) − P(A ∩ B).
f) If A, B, and C are any three events, then P(A ∪ B ∪ C) = P(A) + P(B) + P(C) − P(A ∩ B) −P(A ∩ C) − P(B ∩ C) + P(A ∩ B ∩ C).

2. Conditional Probability

(1) Definition

The conditional probability of an event A, given that event B has occurred, is

defined by $P(A|B) = \frac{P(A \cap B)}{P(B)}$, provided that P(B) > 0.

The probability that two events, A and B, both occur is given by the multiplication

rule, P(A ∩ B) = P(A)P(B|A), provided P(A) > 0 or by P(A ∩ B) = P(B)P(A|B), provided P(B) > 0.

(2) Independent Events
Definition:
Events A and B are independent if and only if P(A∩B) = P(A)P(B). Otherwise, A and B are called dependent events.

Theorem:
If A and B are independent events, then the following pairs of events are also independent:
(a) A and B′;
(b) A′ and B;
(c) A′ and B′.

Mutually Independent:
Events A, B, and C are mutually independent if and only if the following two conditions hold:
(a) A, B, and C are pairwise independent; that is,
P(A ∩ B) = P(A)P(B), P(A ∩ C) = P(A)P(C),
and
P(B ∩ C) = P(B)P(C).
(b) P(A ∩ B ∩ C) = P(A)P(B)P(C).

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3. Bayes' Theorem

If A is an event, then A is the union of m mutually exclusive events, namely,
$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_m \cap A)$.
Thus,

$$P(A) = \sum_{i=1}^{m} P(Bi \cap A)$$
$$= \sum_{i=1}^{m} P(Bi)P(A|Bi),$$

which is sometimes called the **law of total probability**. If $P(A) > 0$, then

$$P(B_k | A) = \frac{P(Bk \cap A)}{P(A)}, \; k = 1, 2, \dots, m.$$

Therefore, we derive Bayes' theorem:

$$P(B_k | A) = \frac{P(Bk)P(A|Bk)}{\sum_{i=1}^{m} P(Bi)P(A|Bi)}, \; k = 1, 2, \dots, m.$$

The conditional probability $P(B_k | A)$ is often called the **posterior probability** of B_k. $P(B_k)$ is often called the **prior probability** of the event B_k.

Reference: Robert V. Hogg, Probability and statistical inference, 9th Ed.